Angular momentum eigenstates of the isotropic 3-D harmonic oscillator: Phase-space distributions and coalescence probabilities

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We have computed the probabilities for coalescence of two distinguishable, non-relativistic particles into a bound state described by a isotropic 3-D harmonic oscillator potential [1]. The initial particles are represented by generic wave packets of given average positions and momenta. We have used a phase-space formulation, and to this end we have recalculated the Wigner distribution functions of angular momentum eigenstates for the isotropic 3-dimensional harmonic oscillator. These distribution functions have been discussed in the literature before [2], but we have utilized an alternative approach to obtain these functions. Along the way, we have derived a general formula that expands angular momentum eigenstates in terms of products of 1-dimensional harmonic oscillator eigenstates.

The expansion of quantum mechanical states in terms of angular momentum eigenstates and vice versa is often an important task. For the isotropic harmonic oscillator this expansion does not seem to be readily available in the literature. We have worked out analytic expressions for the expansion of 3-D oscillator eigenstates (k, l, m) with given radial, angular momentum, and magnetic quantum numbers, in terms of products of 1-D harmonic oscillator states given by quantum numbers (n_x, n_y, n_z). The latter are often the preferred eigenstates to deal with the 3D harmonic oscillator due to the large number of results available for the 1-D harmonic oscillator. As an example, for the case $k = 0$, we find the expansion coefficient to be

$$\langle n_x n_y n_z | 0lm \rangle = \sqrt{\frac{(l+m)!\,(l-m)!}{2^{2l} n_x!\, n_y!\, n_z!\, (2k+2l-1)!}}\, 2^{n_z} i^{n_y} \binom{n_y}{\kappa}\, {}_2F_1(-\kappa, -n_x; 1-\kappa+n_y; -1)$$

where $\kappa = (l + m - n_z)/2$, and ${}_2F_1$ is a hypergeometric function.

The Wigner distributions $W_{kl}(\boldsymbol{r}, \boldsymbol{q})$, for harmonic oscillator states (k, l) averaged over the magnetic quantum number m, are then straight forward to compute from the well-known counterparts in 1-D. Due to the symmetries of the system they only depend on the magnitudes r and q, and on the angle θ between position and momentum vectors $\boldsymbol{r}$ and $\boldsymbol{q}$. See Fig. 1 for an example.

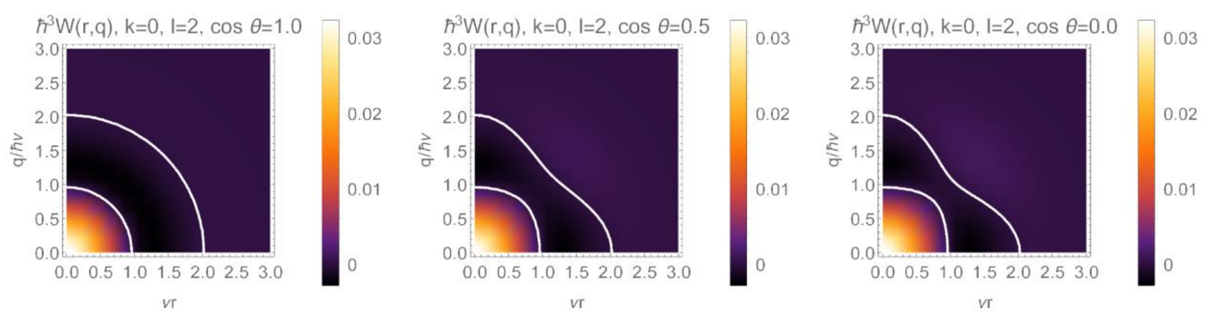

Fig. 1. Wigner distributions $\boldsymbol{W_{kl}(r, q)}$, for the harmonic oscillator state $(\boldsymbol{k = 0, l = 2})$ for three values of the angle $\boldsymbol{\theta}$, as a function of the dimensionless phase space coordinates $\boldsymbol{\nu r}$ and $\boldsymbol{q/\nu\hbar}$, where $\boldsymbol{1/\nu}$ is the typical length scale of the harmonic oscillator. White lines indicate nodal lines of vanishing $\boldsymbol{W_{02}}$.

Finally, we have computed the probabilities of two particles, described quantum mechanically by wave packets, to form a bound state through an isotropic 3-D harmonic oscillator potential between them. We make minimal assumptions about the wave packets, choosing them to be Gaussian and isotropic. The final probabilities $P_{kl}(\boldsymbol{r},\boldsymbol{p})$ for forming a bound state with quantum numbers (k,l) are expressed in terms of the relative center coordinates of the wave packets in phase space, $\boldsymbol{r}$ and $\boldsymbol{p}$. For example, in the simplest case[1]

$$P_{10} = \frac{1}{2}e^{-v}\left(\frac{1}{3}v^2 - \frac{1}{3}t\right)$$

for $k = 1,\ l = 0,$ where $v = (\nu^2 r^2 + p^2/\nu^2\hbar^2)/2$ and $t = (r^2p^2 - (\boldsymbol{r}\cdot\boldsymbol{p})^2)/\hbar^2$. Since $t = L^2/\hbar^2$, where L is the classical angular momentum of the wave packet centroids, one can clearly correlate the initial angular momentum of the 2-quark system with the probabilities to form particular orbital angular momentum states l. See Fig. 2 for an example.

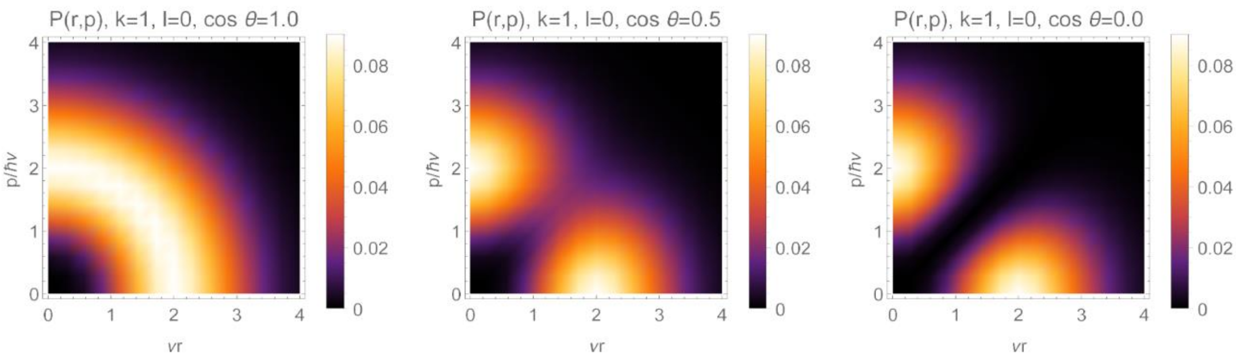

Fig. 2. Probability densities $\boldsymbol{P_{kl}(r,p)}$ to form bound states in a harmonic oscillator state $(\boldsymbol{k=1, l=0})$ for three values of the angle $\boldsymbol{\theta}$ between the relative position vector $\boldsymbol{r}$ and the relative momentum vector $\boldsymbol{p}$ of the two wave packets, as a function of the dimensionless phase space coordinates $\boldsymbol{\nu r}$ and $\boldsymbol{p/\nu\hbar}$. As a radially excited state, which is not orbital angular momentum excited, smaller values of $\boldsymbol{\theta}$ are preferred for $(\boldsymbol{k=1, l=0})$.

Work is under way to apply this formalism to the coalescence of quarks into ground state, excited, and highly excited meson states.

[1] M. Kordell II, R.J. Fries and C.M. Ko, arXiv: 2112.12269, Annals of Physics (Accepted).

[2] S. Shlomo and M. Prakash, Nucl. Phys. **A357**, 157 (1981).

[1] This refers to the fact that results are simplest if the widths of the particle wave packets and the harmonic oscillator potential are in a certain relation [1].